CERTAIN PROPERTIES OF SPECIAL OPERATORS USED IN THE THEORY OF CREEP

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1. Rabotnov [1] has constructed a new special function

$$\ni_\alpha (x, s) = s^{-\alpha} \sum_{\nu=0}^{\infty} x^\nu s^{\nu(1-\alpha)} [\Gamma(\nu - \nu\alpha - \alpha + 1)]^{-1} \qquad (0 < \alpha < 1) \tag{1}$$

which serves as the kernel of an integral operator $\ni_\alpha{}^*$, and has established two fundamental properties of this operator, which make Volterra's principle [1] the most effective means of solving problems in the linear theory of creep of homogeneous and isotropic bodies. This principle can also be successfully employed in solving problems in the theory of creep of nonhomogeneous and anisotropic bodies, if one can establish appropriate properties of the class of $\ni_\alpha{}^*$-operators.

Theorem 1. If $x_1 \neq x_2 \neq \ldots \neq x_m$, then for $m > 1$ we have

$$\prod_{k=1}^{m} \ni_\alpha{}^* (x_k) = \sum_{k=1}^{m} \Big[\prod_{i=1}^{m} (x_k - x_i)\Big]^{-1} \ni_\alpha{}^* (x_k) \qquad (k \neq i) \tag{2}$$

Proof. We can verify (2) by mathematical induction. Indeed, for $m = 2$ the relation (2) coincides with the formula of Rabotnov [1]

$$\ni_\alpha{}^* (x_1) \ni_\alpha{}^* (x_2) = [\ni_\alpha{}^* (x_1) - \ni_\alpha{}^* (x_2)] (x_1 - x_2)^{-1} \tag{3}$$

Let

$$\prod_{k=1}^{m-1} \ni_\alpha{}^* (x_k) = \sum_{k=1}^{m-1} \Big[\prod_{i=1}^{m} (x_k - x_i)\Big]^{-1} \ni_\alpha{}^* (x_k)$$

Then

$$\prod_{k=1}^{m} \ni_\alpha{}^* (x_k) = \ni_\alpha{}^* (x_m) \prod_{k=1}^{m-1} \ni_\alpha{}^* (x_k) =$$

$$= \sum_{k=1}^{m-1} \Big[\prod_{i=1}^{m} (x_k - x_i)\Big]^{-1} [\ni_\alpha{}^* (x_k) - \ni_\alpha{}^* (x_m)] = \sum_{k=1}^{m} \Big[\prod_{i=1}^{m} (x_k - x)\Big]^{-1} \ni_\alpha{}^* (x_k).$$

since

$$\sum_{k=1}^{m-1}\Big[\prod_{i=1}^{m}(x_k - x_i)\Big]^{-1} = -\Big[\prod_{i=1}^{m-1}(x_m - x_i)\Big]^{-1}$$

Thus, the validity of (2) has been proved.

Corollary. For $x_1 \neq x_2 \neq \ldots \neq x_m$ it follows from (2) that

$$\sum_{k=1}^{m} M_k \prod_{n=1}^{k} \ni_\alpha^*(x_n) = \sum_{k=1}^{m}\Big\{\Big[\prod_{i=1}^{k}(x_k - x_i)\Big]^{-1}\sum_{p=k}^{m} M_p\Big\}\ni_\alpha^*(x_k) \tag{4}$$

Remark. When the x_k have the same values, the right-hand sides of (2) and (4) become indeterminate. In order to clarify this indeterminacy it is convenient to rewrite (2) in the following form

$$\prod_{k=1}^{m} \ni_\alpha^*(x_k) = [V_{(m)}(x_1, \ldots, x_m)]^{-1}\sum_{k=1}^{m} A_{k;\, m-1}\ni_\alpha^*(x_k) \tag{5}$$

where $A_{k;\, m-1}$ is the cofactor of the element x_k^{m-1} ($k = 1, \ldots, m$) of the Vandermonde determinant $V_{(m)}(s_1, \ldots, x_m)$. Then for $x_1 = \ldots = x_{j+1}$, where $j + 1 < m$, it follows from (5) that

$$\prod_{k=1}^{m} \ni_\alpha^*(x_k) = \frac{\left[\frac{\partial}{\partial x_j}\Big[\cdots\Big[\frac{\partial}{\partial x_1}\sum_{k=1}^{m} A_{k,\, m-1}\ni_\alpha^*(x_k)\Big]x_1 = x_2\Big]\right]x_j = x_{j+1}}{\left[\frac{\partial}{\partial x_j}\Big[\cdots\Big[\frac{\partial}{\partial x_1}V_{(m)}(x_1, \ldots, x_m)\Big]x_1 = x_2\Big]\right]x_j = x_{j+1}} \tag{6}$$

In (6) the differentiation is performed first with respect to x_1, then with respect to x_2, after substituting x_2 for x_1, and so on up to x_j.

For $x_1 = \ldots = x_{j+1}$ the formula (4) is transformed in a similar fashion.

Theorem 2. If $r_1 \neq r_2 \neq \ldots \neq r_m$, then

$$\Big[1 - \sum_{k=1}^{m} M_k \prod_{n=1}^{k} \ni_\alpha^*(x_n)\Big]^{-1} = 1 + \sum_{k=1}^{m} a_k \ni_\alpha^*(r_k) \tag{7}$$

where r_k ($k = 1, 2, \ldots, m$) are the roots of the equation

$$1 + \sum_{k=1}^{m}\frac{B_k}{x_k - r} = 0 \qquad \Big(B_k = \Big[\prod_{i=1}^{k}(x_k - x_i)\Big]^{-1}\sum_{p=k}^{m} M_p\Big) \tag{8}$$

and the coefficients a_k are determined from the system of linear equations

$$1 + \sum_{k=1}^{m}(B_n - r_k)^{-1} a_k = 0, \qquad B_n \neq r_k \qquad (n = 1, \ldots, m) \tag{9}$$

the determinant of which does not vanish.

Proof: It follows from (7), taking note of (4), that

$$\sum_{k=1}^{m}\left[a_k\ni_\alpha^*(r_k)-B_k\ni_\alpha^*(x_k)\right]-\sum_{k=1}^{m}a_k\ni_\alpha^*(x_k)\sum_{k=1}^{m}B_k\ni_\alpha^*(x_k)=0 \qquad (10)$$

After employing (3) $2m$ times, we obtain from (10)

$$\sum_{k=1}^{m}\left[a_k\left(1+\sum_{n=1}^{m}\frac{B_n}{x_k-r_k}\right)\ni_\alpha^*(r_k)-B_k\left(1+\sum_{n=1}^{m}\frac{a_n}{B_k-r_n}\right)\ni_\alpha^*(x_k)\right]=0 \qquad (11)$$

Since all the a_k and B_k do not vanish simultaneously, it follows from (11) that

$$1+\sum_{k=1}^{m}B_k(x_k-r_n)^{-1}=0, \qquad x_k\neq r_n \qquad (12)$$

$$1+\sum_{k=1}^{m}a_k(B_n-r_k)^{-1}=0, \qquad B_n\neq r_k \qquad (n=1,\ldots,m) \qquad (13)$$

From the form of the m equations (12) it follows that r_n are roots of (5).

The system of equations (13) coincides with (9) and serves to determine the coefficients a_k, which can be expressed in terms of the already known quantities r_k and $B_n (n,\ k = 1,\ \ldots,\ m)$.

Remark. In the case of multiple roots of (8), derivatives of operator appear on the right-hand side of (7), similar to (6), after the indeterminacy is eliminated.

2. As an application of the above, let us examine the problem of the torsion of a shaft made up of m cylindrical layers, rigidly welded along the surfaces of contact, taking into account the creep of each layer. If each layer has cylindrical anisotropy, coinciding with the axis of the shaft, and if there exist planes of hereditary-elastic symmetry, then in order to apply Volterra's principle one must replace the elastic constants $G_{\theta z}{}^k$ in formula (44.13) of Lekhnitskii's book [2], which defines the stresses $\tau_{\theta z}{}^k$ (index $k = 1,\ \ldots,\ m$), by the operators

$$\bar{G}_{\theta z}{}^{kt}=G_{\theta z}{}^{k0}[1-\chi_k\ni_\alpha^*(-\beta_k)] \quad (\beta_k=\tau_k^{\alpha-1},\ \chi_k=\lambda_k\tau_k^{\alpha-1},\ \lambda_k=(G_{\theta z}{}^{k0}-G_{\theta z}{}^{k\infty})/G_{\theta z}^{k0})$$

where τ_k is the relaxation time, and $G_{\theta z}{}^{k0}$ and $G_{\theta z}{}^{k\infty}$ are the instantaneous and the steady state shear moduli of the material in the kth layer. Since $G_{\theta z}{}^{k0} > G_{\theta z}{}^{k\infty}$, it follows that $0 < \lambda_k < 1$. As a result of this change, and after several transformations, we obtain

$$\tau_{\theta z}{}^{kt}=\tau_{\theta z}{}^{k0}[1-\chi_k\ni_\alpha^*(-\beta_k^*)]\left[1-\sum_{k=1}^{m}\chi_k q_k\ni_\alpha^*(-\beta_k)\right]^{-1} \qquad (14)$$

where

$$q_k = G_{\theta z}{}^{k0}(b_k - b_{k-1})\Big[\sum_{k=1}^{m}(b_k - b_{k-1})\,G\quad\Big]^{-1} \tag{15}$$

Here b_k is the distance of the boundary between the kth and $(k+1)$-st layer from the axis ($b_0 = a$, $b_m = b$, where a and b are the inner and outer radii of the shaft cross-section), $\tau_{\theta z}{}^{kt}$ and $\tau_{\theta z}{}^{k0}$ are the effective and the instantaneous stresses. The shaft is deformed by twisting moments M_0 which do not change with time t, and which are applied at the ends of the shaft. In this problem $x_k = -\beta_k$ and $B_k = q_k \chi_k$, where $\beta_k > 0$, $0 < q_k < 1$, $\chi_k < \beta_k$. Therefore, equation (8) is transformed into

$$\sum_{k=0}^{m}\Big[1 - \sum_{i=1}^{m} q_i(\beta_i + \delta_k)^{-1}\Big]\eta_k r^k = 0 \tag{16}$$

Here

$$\delta_0 = 0, \quad \delta_m = 1, \quad \eta_0 = \prod_{i=1}^{m}\beta_i, \quad \eta_k = \sum_{i_1,\ldots,i_k}^{m}\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k} \qquad \begin{matrix}(k=1,\ldots,m-1)\\(i_1 \neq \ldots \neq i_k)\end{matrix}$$

$$\delta_k = \eta_k\eta_{k-1}^{-1}, \quad \eta_{k-1} = \sum_{i_1,\ldots,i_{k-1}}^{m}\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{k-1}} \quad (i_1 \neq \ldots \neq i_{k-1} \neq \ldots \neq i)$$

The structure of the coefficients of (16) is such that the inequality

$$\lambda_1 q_1 + \cdots + \lambda_m q_m \leqslant 1 \tag{17}$$

is a sufficient condition for the positiveness of all the coefficients of (16). A condition necessary to satisfy (17) always holds, since $0 < \lambda_i q_i < 1$. For $m = 1$ (the case of a homogeneous shaft) we have $\lambda_1 q_1 < 1$, which gives $r < 0$; for $m = 2$ the inequality (17) is sufficient to guarantee that the real parts of the roots of (16) be negative; for $m > 2$ we must impose the Hurwitz conditions in addition to (17), for all the roots of (16) to have the character indicated above.

If r_k are simple roots of (16), then it follows from (14) that

$$\tau_{\theta z}{}^{kt} = \tau_{\theta z}{}^{k0}\Big\{1 - \int_0^t \sum_{k=1}^{m}\Big[\theta_1 \ni_\alpha(-\beta_k, s) - \theta_2 \ni_\alpha(r_k, s)\Big]ds\Big\} \tag{18}$$

where

$$\theta_1 = \chi_k[1 - a_k(\beta_k + r_k)^{-1}], \qquad \theta_2 = a_k[1 - \chi_k(\beta_k + r_k)^{-1}]$$

Here the coefficients a_k are determined from (8) with $B_k = \chi_k q_k$. The expression (18) is a series converging for any t, since the $\ni_\alpha$-function is defined by the uniformly converging series (1). Since the series (18) converges slowly, it is reasonable to approximate $\tau_{\theta z}{}^{kt}$ in the form

$$\tau_{\theta z}{}^{kt} \approx \tau_{\theta z}{}^{k0}\left\{1-\sum_{k=1}^{m}\left[\frac{\theta_1}{\beta_k}\left(1-\exp\left(-\gamma\beta_k t^{1-\alpha}\right)\right)+\frac{\theta_2}{r_k}\left(1-\exp\left(\gamma r_k t^{1-\alpha}\right)\right)\right]\right\} \qquad (19)$$

where $\gamma = (1-\alpha)^{1-\alpha}$. The approximate equation (19) becomes exact for $\alpha = 0$.

Consequently, if all the shaft layers have the same creep characteristics, $\tau_{\theta z}{}^{kt}$ can change in one way only as $t \to \infty$, namely, $\tau_{\theta z}{}^{kt}$ must approach a finite limit monotonically.

If the creep characteristics of the shaft layers differ from each other, then the approach of $\tau_{\theta z}{}^{kt}$ to a steady (limiting) state as $t \to \infty$ can take place by means of a monotonic, nonmonotonic, and a retarded change of $\tau_{\theta z}{}^{kt}$ with time, depending on the character of the roots of (16). The case where there is no finite limiting value for $\tau_{\theta z}{}^{kt}$ as $t \to \infty$ is also possible.

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